General Pappus

Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve with length L. Consider the solid S which has sections perpendicular to γ and to the plane that are the same figure of area A, with the point of intersection with γ always a fixed point on the planar section. We will calculate the volume of S.

Let $(u(s), v(s), 0)$ be an arclength parametrization of γ, traced counterclockwise. The transversal section can be parametrized by $r, t \in D \subset \mathbb{R}^2$, with $r = t = 0$ representing the coordinates of the point of intersection with γ. With this, the region S is parametrized by

$$P = (x, y, z) = (u(s), v(s), 0) + r\hat{n}(s) + t(0, 0, 1) \;,\; s \in [0, L] \,,\; r, t \in R \;,$$

where $\hat{n}(s) = (v'(s), -u'(s), 0)$ is the outward pointing unit normal to the curve. A simple calculation shows that the jacobian $\partial(x, y, z)/\partial(s, r, t)$ is given by

$$1 + rk(s) \,,$$

where $k(s)$ is the signed curvature γ with respect to $-\hat{n}$. Indeed

$$P_s = \hat{t} + r\hat{n}_s = \hat{t} + rk\hat{t} = (1 + kr)\hat{t} \,,$$

$$P_r = \hat{n} \,,$$

$$P_t = (0, 0, 1) \,.$$

After integrating the jacobian we obtain

$$V = LA + 2\pi \iint_R r dr dt \,.$$

When the r component of the center of gravity of R vanishes, we obtain a generalization of the theorem of Pappus.